INVARIANTS AND CANONICAL FORMS

By E. J. Wilczynski

Department of Mathematics, University of Chicago

Communicated by E. H. Moore, August 7, 1918

Every student of the theory of invariants has observed the fact that the coefficients of a unique canonical form are invariants. But a general *a priori* proof of this fact, sufficiently general to cover all of the cases needed in the applications, seems to be lacking. It is the purpose of this paper to furnish such a proof, making use only of the abstract principles which are common to all known invariant theories.

We begin by giving a brief outline of some of these invariant theories, so that we may have these instances in mind when we formulate our general theory. Consider first a binary n-ic,

$$(p) = \sum_{i=0}^{n} p_i x_1^i x_2^{n-i}. \tag{A}$$

This binary *form* is a function of $p_0, p_1, .., p_n$ and of x_1, x_2. The p's are called the *coefficients*, and the x's the *variables* of the form. We introduce new variables by putting

$$\bar{x}_i = \alpha_{i1} x_1 + \alpha_{i2} x_2 \quad ,(i = 1, 2),$$

where the quantities α_{ik} are arbitrary constants with a non-vanishing determinant, thus transforming the form (p) into a new form $(\bar{p})$. Those combinations of the coefficients p_k which are equal to the same functions of the coefficients $\bar{p}_k$ are called *invariants* of the form. In the classical theory of invariants it is really the *equation* $(p) = 0$ which is the object of study rather than the *form* or *function* (p). Consequently the additional transformations, operating upon the coefficients only, $\bar{p}_i = \lambda p_i$, $(i = 0, 1, 2, .., n)$, where λ is an arbitrary constant, are introduced. An invariant of the *equation* must remain unaltered by these transformations also.

A second invariant theory is concerned with the class of linear differential expressions or forms

$$p_0 \frac{d^n y}{dx^n} + p_1 \frac{d^{n-1} y}{dx^{n-1}} + \; . \; . \; . \; . + p_n y, \tag{B}$$

where $p_0, p_1 \ldots, p_n$, and y are functions of x. The *coefficients* p_k in this case are functions of x, whereas in instance (A) they are numbers, real or complex. In instance (A) the variables x_1, x_2 are also numbers; in (B) the variables $y, dy/dx, \ldots, d^n y/dx^n$ are functions. In this case we may even think of y as the *only* variable, since the variables dy/dx, etc., are determined when y is given.

Let us now transform the variable y by putting $y = \lambda(x)\bar{y}$, where $\lambda(x)$ is an arbitrary function. Then (B) goes over into a new differential form $(\bar{B})$

whose coefficients $\bar{p}_k$ depend upon $p_0, p_1, \ldots, p_n$ and λ. The corresponding invariants are of considerable importance. Here also we must distinguish between the theory of invariants of forms, and the theory of invariants of equations.

As a third instance, let us consider the class of analytic functions

$$f(x) = p_0 + p_1 x + p_2 x^2 + \ldots + p_n x^n + \ldots, \tag{C}$$

where the coefficients $p_0, p_1 \ldots, p_n, \ldots,$ and the variable x are complex numbers. We introduce transformations of the form

$$\bar{x} = \frac{\alpha x + \beta}{\gamma x + \delta}, \quad \alpha\delta - \beta\gamma \neq 0,$$

where α, β, γ, δ are arbitrary constants, and attempt to find invariants of $f(x)$ under all such transformations. In this case we are dealing with a form which has infinitely many coefficients, forming a denumerable set.

We shall list one further instance of our general theory. Let $K(x, \xi)$ be a real continuous function of x and ξ in the region $0 \leqq x \leqq 1$, $0 \leqq \xi \leqq 1$, and let $\varphi(\xi)$ be a real continuous function of ξ in the interval $0 \leqq \xi \leqq 1$. Then

$$I(x) = \int_0^1 K(x, \xi)\,\varphi(\xi)\,d\xi \tag{D}$$

may be regarded as a form whose value, as function of x, depends upon the choice of the K and φ functions. We may think of the functions of x to which $K(x, \xi)$ reduces, for all special values of ξ between 0 and 1, as the coefficients of the form. Thus the number of these coefficients is non-denumerable, even if the function $K(x, \xi)$ be regarded as given. As to the variables of the form $I(x)$ we still have two choices. We may think of the *function* $\varphi(\xi)$ as ranging over the class of all continuous functions and regard $\varphi(\xi)$ as the only variable of the form; or else we may consider the *functional values* of $\varphi(\xi)$ as the variables. In the latter case, the *range* of the variables would be the class of real numbers, and the number of variables would be continuously infinite.

We may transform (D) by putting

$$\varphi(\xi) = \lambda(\xi)\,\bar{\varphi}(\xi),$$

where $\lambda(\xi)$ is an arbitrary continuous function of ξ. This will transform $I(x)$ into $\bar{I}(x)$, where

$$\bar{I}(x) = \int_0^1 \bar{K}(x, \xi)\,\bar{\varphi}(\xi)\,d\xi, \quad \bar{K}(x, \xi) = K(x, \xi)\,\lambda(\xi).$$

In particular it will be possible to choose $\lambda(\xi)$ in such a way as to make

$$K(\tfrac{1}{2}, \xi) = 1$$

except for those values of ξ for which $K(\frac{1}{2}, \xi) = 0$; the resulting integral may be said to be in its *canonical form*.

For the purposes of our general theory, which shall include all of the instances mentioned as well as infinitely many others, *we postulate a class* $[F]$ *of forms, or functions of two general arguments;* $F(p, x)$. As the notation indicates, the arguments of such a form are of two kinds, the *coefficients* p, and the *variables* x. Both p and x are supposed to be *general variables* in the sense of E. H Moore, each argument varying on its own range. This range may be a continuous or discrete class of numbers, or a class of functions, or any other well-defined range; it may be the same for p and x, or different Finally the range over which F varies may be different from either or both of the ranges of p or x. In the language of ordinary analysis this general formulation includes single forms or systems of forms, whose coefficients and variables may be finite or infinite in number, and in the latter case denumerably or non-denumerably infinite.

We further assume that *the class of forms* $[F]$ *is well-defined* This means that a criterion 's given by means of which we may decide whether a given form does or does not belong to $[F]$. This criterion will also enable us to distinguish between the coefficients and the variables of the form

Two forms of the class $[F]$ *are said to be identical if their corresponding coefficients are equal.* Equality of the correspondng variables is *not* required for the identity of two forms. For this reason it is frequently convenient to suppress the notation for the variables in the symbol for a form, and to use the simpler symbol $F = (p)$ to replace $F(p, x)$.

We postulate, in the second place, a group of transformations, which operates upon the variables of a form F *of* $[F]$ *and transforms every* F *of* $[F]$ *into another form* $\overline{F}$ *of the same class. The coefficients* $\overline{p}$ *of* $\overline{F}$ *will depend on the coefficients* p *of* F *and upon elements which occur in the transformation used. We assume that we obtain in this way a new group* G *operating upon the coefficients of the form.* G *is said to be induced by* g.

In this postulate the word group is uded in the usual sense. Thus we call a set of operations a group if the identity is included among its operations, if the operations possess an associative law of combination, if the product of any two operations of the set and also the inverse of every operation of the set belong to the set. The two groups, g and G, may however contain a finite number of operations or infinitely many; in the latter case they may be discrete or continuous; if they are continuous, they may be finite or infinite, in the sense of Lie.

By an appropriate modification of the group G, *we may include in our theory not only the invariants of forms, but also the theory of invariants of equations and systems of equations.* This has been pointed out already in connection with our preliminary discussion of instance (A).

If there exists at least one transformation in the group G *which, when applied to a form* F_1 *of* $[F]$, *transforms* F_1 *into a form* F_2, *the two forms* F_1 *ad* F_2 *are said to be equivalent under the group* G.

Let us now define a proper sub-class $[\Phi]$, of $[F]$, by imposing some property upon the coefficients p which is not satisfied by all forms of $[F]$. It is assumed that this property is well defined, so that we may be able to decide whether a given form F, of $[F]$, does or does not belong to the sub-class $[\Phi]$. Let us assume that, for every form F of $[F]$, there exists in G at least one transformation S which transforms F into an equivalent form of the sub-class $[\Phi]$. *We shall then say that* $\Phi = S[F]$ *is a canonical form of* F. It may happen that such a canonical form equivalent to F under the group G, and belonging to the sub-class $[\Phi]$ exists merely for those forms of F which do not belong to a well-defined sub-class $[\Psi]$ of $[F]$. We shall speak of the forms of the sub-class $[\Psi]$ as *exceptional* forms, and call all the other forms of $[F]$ *generic* forms. Of course the term, *general* form of $[F]$, includes both the exceptional and the generic forms. Under these circumstances we shall still speak of Φ as a canonical form of F, but we shall add the qualifying phrase *for the generic case* whenever the distinction becomes necessary.

In general there will be many transformations of G which transform every generic F into a canonical form of the sub-class $[\Phi]$. If all of these transformations transform every generic F into the same form Φ, of the sub-class $[\Phi]$, we shall say that *the canonical form* Φ *is unique.*

It remains to define the term *invariant.* A function I, of the coefficients p of a form F, is an absolute invariant under the broup G, if it is equal to the same function of the corresponding coefficients of any form $\bar{F}$ which is equivalent to F, by means of a transformation of the group G.

This notion may be regarded as including also the notion *covariant.* For, we may replace the given form F by another form, or system of forms, who coefficients depend also upon the *variables* of F.

We are now ready to prove the following theorem.

Let $[F]$ *be a well-defined class of forms, and let* $[\Phi]$ *be a proper sub-class of* $[F]$. *Let* G *be a group of transformations which transforms every form of* $[F]$ *into a form of the same class. By a generic form of* $[F]$ *we mean one which is equivalent to a form of* $[\Phi]$ *under* G. *Then, there exists, for every generic form of* $[F]$ *at least one transformation in* G *which transforms* F *into a canonical form of the sub-class* $[\Phi]$. *If this canonical form is unique, its coefficients are one-valued absolute invariants of the form* F *for the group* G.

Proof.—Let F be a generic form of $[F]$, and let Φ be its canonical form. Let S be the most general operation of G which transforms F into Φ. We shall have symbolically

$$S\,(F) = \Phi. \tag{1}$$

The operation S will depend, in general, upon the coefficients p, of F, and may contain, besides, arbitrary elements in great number. The canonical form Φ belongs to the sub-class $[\Phi]$ of $[F]$. Since this canonical form is, by hypothesis, unique, its coefficients π are independent of the arbitrary elements which may occur in S, and they are one-valued functions of the coefficients

of F in the sense, that when F is given, the coefficients of Φ are determined uniquely. We may express this symbolically by the equation

$$\pi = U(p), \tag{2}$$

where $U(p)$ is the symbol for a one-valued function of the coefficients p.

Now let T be any transformation of G, and let

$$\bar{F} = T(\bar{F})$$

be any form of $[F]$ equivalent to F under G. Since we assumed that F was generic, and since

$$F = T^{-1}(F),$$

we conclude that $\bar{F}$ is also generic. In fact we find

$$S(F) = ST^{-1}(F) = \Phi,$$

so that ST^{-1} is an operation of G which transforms $\bar{F}$ into a canonical form of the sub-class $[\Phi]$.

Let $\bar{S}$ be the most general operation of G which transforms $\bar{F}$ into a canonical form of the sub-class $[\Phi]$, so that

$$\bar{S}(\bar{F}) = \bar{\Phi},$$

and let $\bar{\pi}$ denote the coefficients of $\bar{\Phi}$. These coefficients will depend on the coefficients $\bar{p}$ of $\bar{F}$, in exactly the same way as the coefficients π of Φ depend upon the coefficients p of F. That is, we shall have, in a manner analogous to (2),

$$\bar{\pi} = U(\bar{p}). \tag{3}$$

We also know that

$$\bar{\Phi} = \Phi;$$

for we have seen already that ST^{-1} will transform $\bar{F}$ into Φ, and we are assuming that every generic form $\bar{F}$ of $[F]$ has a *unique* canonical form of the sub-class $[\Phi]$.

But, if two forms of a class are equal, their corresponding coefficients are equal. Therefore we have $\bar{\pi} = \pi$ or, according to (2) and (3).

$$U(\bar{p}) = U(p).$$

In other words the coefficients of Φ are indeed absolute invariants of F under the group G, and therefore our theorem is demonstrated.

If the forms of the class $[F]$, which are generic forms from the point of view of the canonical form chosen, do not constitute the whole of $[F]$ there will remain in $[F]$ certain exceptional forms constituting a sub-class $[H]$ of $[F]$. But the theorem may be applied to these exceptional forms as well, whenever a unique canonical form exists for a generic one of the exceptional forms; but of course, the canonical form in this case will not belong to the class $[\Phi]$, but

to a certain sub-class $[X]$, of $[H]$. If the forms of $[H]$ are not all generic from this new point of view, we may adopt a new canonical form for the exceptional ones, and continue in this way.

We have observed that the coefficients of a unique canonical form are absolute invariants, and moreover one-valued invariants, in the sense, that their values are uniquely determined as soon as the coefficients of the form F are given. In the ordinary theory of algebraic invariants, it is at once apparent that these invariants are algebraic functions of the coefficients of F. Consequently it follows from their one-valuedness that, in this case, these invariants are *rational* functions of the coefficients. In the theory of invariants of linear differential equations, the uniquenes of a canonical form gives rise to invariants which are rational functions of the coefficients of the differential equation and of their derivatives.

There are many cases in which a k-valued canonical form is obtained rather than a unique one. That is, if we resume our terminology, the sub-class $[\Phi]$ contains not merely one, but exactly k forms $\Phi_1, \Phi_2, \ldots, \Phi_k$, each of which is equivalent to F by a transformation of the group G. *The coefficients of these canonical forms will still be absolute invariants of F, but they will be k-valued functions of its coefficients.* It is obviously possible to find an equation of degree k with one-valued invariants as coefficients, of which these k-valued invariants are the roots. In the theory of algebraic invariants we obtain in this way irrational invariants, as roots of an equation whose coefficients are rational invariants.

TYPES OF PHOSPHORESCENCE

By Edward L. Nichols and H. L. Howes

Department of Physics, Cornell University

Communicated, August 1, 1918

The existence of phosphorescence of exceedingly short duration was long ago revealed by the phosphoroscope of Becquerel but, until very recently the afterglow of luminescent bodies has been studied quantitatively, only where it is of comparatively long duration. Curves of decay were supposed to be all of the same character. It was assumed that the law of diminution of brightness, as expressed by the equation

$$I = \frac{1}{(a_1 + b_1 t)^2} + \frac{1}{(a_2 + b_2 t)^2}$$

was of general application and that the phosphorescence of various substances differed only in color, brightness and duration.

The measurements of Waggoner[1] and of Zeller[2] on phosphorescence of short duration tended to confirm this view. On the other hand the observations of